

Multiple summing operators on Banach spaces[☆]

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Abstract

In this paper, we improve some previous results about multiple p -summing multilinear operators by showing that every multilinear form from \mathcal{L}_1 spaces is multiple p -summing for $1 \leq p \leq 2$. The proof is based on the existence of a predual for the Banach space of multiple p -summing multilinear forms. We also show the failure of the inclusion theorem in this class of operators and improve some results of Y. Meléndez and A. Tonge about dominated multilinear operators.

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1. Introduction and notation

Motivated by the importance of the theory of absolutely summing linear operators, there have been some attempts to generalize this concept and the related results and tools to the multilinear setting. Perhaps the most important one was initiated by Pietsch in [15], where he introduced the r -dominated multilinear mappings. We say that a multilinear operator $T : X_1 \times \cdots \times X_n \rightarrow Y$ is r -dominated ($1 \leq r < \infty$) if there exists a constant $K > 0$ such that

$$\left(\sum_{i=1}^m \|T(x_i^1, \dots, x_i^n)\|^{r/n} \right)^{n/r} \leq K \prod_{j=1}^n \|(x_i^j)_{i=1}^m\|_r^\omega \quad (1)$$

for all choices of $m \in \mathcal{N}$ and $x_1^j, \dots, x_m^j \in X_j$.

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The class of r -dominated multilinear operators from $X_1 \times \cdots \times X_n$ to Y is a quasi-Banach space with the quasi-norm $\delta_r(T) = \inf\{K : K \text{ verifies (1)}\}$.

The importance of this class arises since these operators verify a domination theorem similar to the linear case. In fact we have the following

Theorem 1.1 [12, Theorem 3.2]. *A multilinear operator $T : X_1 \times \cdots \times X_n \rightarrow Y$ is r -dominated if and only if there exist a constant $K > 0$ and regular probability measures $\mu_j \in C(B_{X_j^*})^*$ ($1 \leq j \leq n$) such that*

$$\|T(x_1, \dots, x_n)\| \leq K \prod_{j=1}^n \left(\int_{B_{X_j^*}} |x_j^*(x_j)|^r d\mu_j(x_j^*) \right)^{1/r} \quad (2)$$

for every $x_j \in X_j$. Moreover, in that case $\delta_r(T) = \min\{K : K \text{ verifies (2)}\}$.

The interested reader can consult [4], [12], or [13] and references therein to know more about this class of operators. To compare our Corollary 3.12 with [13, Theorem 3] we have to point out that δ_r is called $\pi_{r,r}$ in [13].

Recently, Bombal and both authors in [3] and [14], and Matos in [11] have defined and studied the class of multiple summing multilinear operators (although the origin of this class goes back to [16]). This class behaves better in many ways than previous definitions of p -summing multilinear operators, and seems to be the “right” generalization of the linear case for many applications. In fact, it is the main tool used in [3,14] to improve some previous results relating tensor products [14, Proposition 3.3], matrix inequalities [3, Corollary 4.4], and polynomial bounds [3, Corollary 4.4].

In this paper, we present a considerable improvement to [3, Section 5] by showing that, for $1 \leq p \leq 2$, every multilinear form from \mathcal{L}_1 spaces is multiple p -summing. This result is essentially contained in [17] but our approach is different and much shorter. Our main tool is the definition of a predual for the space of multiple p -summing multilinear forms. Moreover, we relate the class of multiple p -summing operators to the class of r -dominated operators and use this relation to improve some results of [13].

The notations and terminology used along the paper will be the standard in Banach space theory, as for instance in [7], which is our main source for unexplained notation. This book is also our main reference for basic facts and definitions concerning most of the topics in this paper. However, before going any further, we shall establish some terminology: X_i, Y will always be Banach spaces and H will stand for a Hilbert space. $\mathcal{L}(X, Y)$ will denote the Banach space of linear bounded mappings from X to Y . For $k \geq 2$, $\mathcal{L}^k(X_1, \dots, X_k; Y)$ will be the Banach space of all the continuous k -linear mappings from $X_1 \times \cdots \times X_k$ into Y . When $Y = \mathbb{K}$ we will omit it and, from now on, ‘operator’ will mean linear or multilinear ‘continuous mapping.’ As usual, $X_1 \otimes_\pi \cdots \otimes_\pi X_n$ stands for the projective tensor product of the Banach spaces X_1, \dots, X_n . Given a Banach space X , X^* stands for its topological dual and B_X denotes its unit ball.

Given a space X and $1 \leq p < \infty$, we say that a sequence $(x_n)_n \subset X$ is strongly p -summable if

$$(\|x_n\|)_n \in \ell_p.$$

We denote by $\ell_p(X)$ the Banach space of all such sequences endowed with the norm

$$\|(x_n)_n\|_p = \left(\sum_n \|x_n\|^p \right)^{1/p}.$$

We say that $(x_n)_n$ is weakly p -summable if, for every $x^* \in X^*$, $(\langle x^*, x_n \rangle)_n \in \ell_p$. We write

$$\|(x_n)_n\|_p^\omega = \sup \left\{ \left(\sum_n \langle x^*, x_n \rangle^p \right)^{1/p} : x^* \in B_{X^*} \right\}.$$

Given $1 \leq p < \infty$, we write $\Pi_p(X; Y)$ for the Banach space of p -summing operators from X into Y . Given $T \in \Pi_p(X; Y)$, $\pi_p(T)$ denotes its p -summing norm.

Let $1 \leq p \leq \infty$ and $\lambda > 1$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if, for every finite dimensional subspace $E \subset X$ there exists another finite dimensional subspace F , with $E \subset F \subset X$ and such that there exists an isomorphism $v: F \rightarrow \ell_p^{\dim F}$ with $\|v\| \|v^{-1}\| < \lambda$. We say that X is an \mathcal{L}_p space if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda > 1$. Clearly, $L_p(\mu)$ is the basic example of an \mathcal{L}_p -space.

Given $n, m_1, \dots, m_n \in \mathbb{N}$, $(x_{i_1, \dots, i_n})_{i_1, \dots, i_n=1}^{m_1, \dots, m_n}$ denotes a multiindex sequence with the index i_j varying from 1 to m_j ($1 \leq j \leq n$). $\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} x_{i_1, \dots, i_n}$ means the same as $\sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} x_{i_1, \dots, i_n}$.

For $n \in \mathbb{N}$ we define the Rademacher functions $r_n: [0, 1] \rightarrow \mathbb{R}$ as $r_n(t) = \text{sign}(\sin 2^n \pi t)$. A Banach space X is said to have cotype q if there is a constant $K > 0$ such that, no matter how we select finitely many vectors $x_1, \dots, x_m \in X$,

$$\left(\sum_{i=1}^m \|x_i\|^q \right)^{1/q} \leq K \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\|^2 dt \right)^{1/2}.$$

The smallest of these constants will then be denoted by $C_q(X)$.

By Kahane's inequality [7, Theorem 11.1], we know that a Banach space X has cotype q if and only if there exists a constant K' such that

$$\left(\sum_{i=1}^m \|x_i\|^q \right)^{1/q} \leq K' \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\|^q dt \right)^{1/q}$$

for every $x_1, \dots, x_m \in X$.

We will call $c_q(X)$ the smallest of these constants. It is trivial to see that $c_q(X) \leq C_q(X)$.

2. Definition and known facts

We recall now our definition.

Definition 2.1. Let $1 \leq p_1, \dots, p_n \leq q < +\infty$. A multilinear operator

$$T: X_1 \times \dots \times X_n \rightarrow Y$$

is *multiple* $(q; p_1, \dots, p_n)$ -summing if there exists a constant $K > 0$ such that, for every choice of sequences $(x_{ij}^j)_{i_j=1}^{m_j} \subset X_j$ the following relation holds:

$$\left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^q \right)^{1/q} \leq K \prod_{j=1}^n \|(x_{ij}^j)_{i_j=1}^{m_j}\|_{p_j}^\omega. \quad (3)$$

In that case, we define the *multiple* $(q; p_1, \dots, p_n)$ -summing norm of T by $\pi_{(q; p_1, \dots, p_n)}(T) = \min\{K : K \text{ verifies (3)}\}$.

The class $\Pi_{(q; p_1, \dots, p_n)}^n(X_1, \dots, X_n; Y)$ of multiple $(q; p_1, \dots, p_n)$ -summing multilinear operators is easily seen to be a Banach space with its norm $\pi_{(q; p_1, \dots, p_n)}$.

A multiple $(q; p, \dots, p)$ -summing operator will be called *multiple* (q, p) -summing and we write $\pi_{(q, p)}$, $\Pi_{(q, p)}^n$ for the associated norm and class, respectively. Moreover, a multiple (p, p) -summing operator will be called *multiple* p -summing and we write again π_p , Π_p^n for the associated norm and class. Clearly, when $n = 1$ we recover the linear definitions.

As mentioned in [3], we get an equivalent definition if we choose infinite weakly summable sequences.

It can be easily proved (as [11, Proposition 2.5]) that

Proposition 2.2. *Let $T : X_1 \times \dots \times X_n \rightarrow Y$ be a multilinear operator, let $1 \leq k \leq n - 1$ and let $T_k : X_1 \times \dots \times X_k \rightarrow \mathcal{L}^{n-k}(X_{k+1}, \dots, X_n; Y)$ be the associated k -linear operator. If $T_k \in \Pi_{(q; p_1, \dots, p_k)}^k(X_1, \dots, X_k; \Pi_{(q; p_{k+1}, \dots, p_n)}^{n-k}(X_{k+1}, \dots, X_n; Y))$, then $T \in \Pi_{(q; p_1, \dots, p_n)}^n(X_1, \dots, X_n; Y)$ and $\pi_{(q; p_1, \dots, p_n)}(T) \leq \pi_{(q; p_1, \dots, p_k)}(T_k)$.*

We showed in [14] that the converse implication does not hold. However, it follows from [14] and [11] that the converse is true when $p = q_1 = \dots = q_n = 1$ and all the X_j are $C(K)$ spaces, or when $p = q_1 = \dots = q_n = 2$ and all the X_j and Y are Hilbert spaces.

3. The results

Let us show first that the multiple p -summing operators are a dual space, provided the image space is a dual (in particular, multiple p -summing multilinear forms are always a dual space). We define in $X_1 \otimes \dots \otimes X_n \otimes Y$ the norm

$$\tilde{\alpha}_p(u) = \inf \left\{ \sum_{m=1}^M \|(y_{m, i_m^1, \dots, i_m^n})_{i_m^1, \dots, i_m^n=1}^{I_m^1, \dots, I_m^n}\|_{p'} \|(x_{m, i_m^1}^1)_{i_m^1=1}^{I_m^1}\|_p^\omega \cdots \|(x_{m, i_m^n}^n)_{i_m^n=1}^{I_m^n}\|_p^\omega \right\},$$

where $1/p + 1/p' = 1$ and the infimum is taken among all the representations

$$u = \sum_{m=1}^M \sum_{i_m^1, \dots, i_m^n=1}^{I_m^1, \dots, I_m^n} x_{m, i_m^1}^1 \otimes \dots \otimes x_{m, i_m^n}^n \otimes y_{m, i_m^1, \dots, i_m^n}.$$

When $Y = \mathbb{K}$ we identify $X_1 \otimes \dots \otimes X_n$ with $X_1 \otimes \dots \otimes X_n \otimes \mathbb{K}$. Then, we will simply denote α_p for the corresponding norm $\tilde{\alpha}_p$ in $X_1 \otimes \dots \otimes X_n$.

Proposition 3.1. *We have that $\tilde{\alpha}_p$ is a tensor norm of order $n + 1$ (in the sense of [9]) such that $(X_1 \otimes \cdots \otimes X_n \otimes Y, \tilde{\alpha}_p)^*$ is isometrically isomorphic to $\Pi_p^n(X_1, \dots, X_n; Y^*)$.*

Proof. For simplicity we write the proof for the case of bilinear operators $X \times Y \rightarrow Z^*$, but our reasonings extend without further complications to the case of more spaces.

It is completely trivial to show that $\tilde{\alpha}_p$ is a norm with the metric mapping property that verifies $\tilde{\alpha}_p \leq \pi$, with π the projective norm. To see that $\tilde{\alpha}_p \geq \epsilon$, where ϵ is the injective norm, all we have to do is to use Hölder's inequality.

Let us consider now $T \in \Pi_p^2(X, Y; Z^*)$ and $\epsilon > 0$. We write \tilde{T} for the associated linear form $\tilde{T} : X \otimes Y \otimes Z \rightarrow \mathbb{K}$. For $u \in X \otimes Y \otimes Z$, we consider a representation

$$u = \sum_{m=1}^M \sum_{i_m, j_m=1}^{I_m, J_m} x_{m, i_m} \otimes y_{m, j_m} \otimes z_{m, i_m, j_m}$$

such that

$$\tilde{\alpha}_p(u) + \epsilon \geq \sum_{m=1}^M \|(x_{m, i_m})_{i_m=1}^{I_m}\|_p^\omega \|(y_{m, j_m})_{j_m=1}^{J_m}\|_p^\omega \|(z_{m, i_m, j_m})_{i_m, j_m=1}^{I_m, J_m}\|_{p'}^\omega.$$

Using Hölder's inequality and the definition of multiple p -summing multilinear operator, we get

$$\begin{aligned} |\tilde{T}(u)| &\leq \pi_p(T) \sum_{m=1}^M \|(x_{m, i_m})_{i_m=1}^{I_m}\|_p^\omega \|(y_{m, j_m})_{j_m=1}^{J_m}\|_p^\omega \|(z_{m, i_m, j_m})_{i_m, j_m=1}^{I_m, J_m}\|_{p'}^\omega \\ &\leq \pi_p(T) (\tilde{\alpha}_p(u) + \epsilon). \end{aligned}$$

For the converse, we consider $S \in (X \otimes Y \otimes Z, \tilde{\alpha}_p)^*$ and $\epsilon > 0$. We denote by $\check{S} : X \times Y \rightarrow Z^*$ the associated bilinear operator. If we take sequences $(x_i)_{i=1}^m \subset X$, $(y_j)_{j=1}^n \subset Y$, we know that there exist sequences $(z_{i,j})_{i,j=1}^{m,n} \subset BZ$, $(\lambda_{i,j})_{i,j=1}^{m,n} \subset \mathbb{K}$ such that $\sum_{i,j=1}^{m,n} |\lambda_{i,j}|^{p'} \leq 1$ and such that

$$\begin{aligned} \left(\sum_{i,j=1}^{m,n} \|\check{S}(x_i, y_j)\|^p \right)^{1/p} - \epsilon &\leq \left(\sum_{i,j=1}^{m,n} |S(x_i \otimes y_j \otimes z_{i,j})|^p \right)^{1/p} \\ &= S \left(\sum_{i,j=1}^{m,n} x_i \otimes y_j \otimes (\lambda_{i,j} z_{i,j}) \right) \leq \|S\| \tilde{\alpha}_p \left(\sum_{i,j=1}^{m,n} x_i \otimes y_j \otimes (\lambda_{i,j} z_{i,j}) \right) \\ &\leq \|S\| \|(x_i)_{i=1}^m\|_p^\omega \|(y_j)_{j=1}^n\|_p^\omega, \end{aligned}$$

just because $\|(\lambda_{i,j} z_{i,j})_{i,j=1}^{m,n}\|_{p'} \leq 1$. \square

Remark 3.2. In [11] and independently, Matos defines a quasi-norm ρ_p in $X_1 \otimes \cdots \otimes X_n \otimes Y$ such that $(X_1 \otimes \cdots \otimes X_n \otimes Y, \rho_p)^* = \Pi_p^n(X_1, \dots, X_n; Y^*)$.

From now on, we denote $(X_1 \otimes \cdots \otimes X_n, \alpha_p)$ by $X_1 \otimes_{\alpha_p} \cdots \otimes_{\alpha_p} X_n$ and we denote its completion by $X_1 \hat{\otimes}_{\alpha_p} \cdots \hat{\otimes}_{\alpha_p} X_n$.

Lemma 3.3. (i) A linear operator $S: X_1 \otimes \cdots \otimes X_n \rightarrow Y$ is α_p -continuous if and only if y^*S is α_p continuous for every $y^* \in B_{Y^*}$. Moreover, in that case

$$\|S\|_{\mathcal{L}(X_1 \hat{\otimes}_{\alpha_p} \cdots \hat{\otimes}_{\alpha_p} X_n, Y)} = \sup_{\|y^*\| \leq 1} \|y^*S\|_{(X_1 \hat{\otimes}_{\alpha_p} \cdots \hat{\otimes}_{\alpha_p} X_n)^*}.$$

(ii) A multilinear operator $T: X_1 \times \cdots \times X_n \rightarrow Y$ is multiple p -summing if its associated linear operator $\bar{T}: X_1 \otimes \cdots \otimes X_n \rightarrow Y$ is α_p continuous and p -summing as an operator $\bar{T}: X_1 \hat{\otimes}_{\alpha_p} \cdots \hat{\otimes}_{\alpha_p} X_n \rightarrow Y$. Moreover, in that case we have that $\pi_p(T) \leq \pi_p(\bar{T})$.

Proof. (i) is a straightforward application of the closed graph theorem. To see (ii), we consider, for $1 \leq j \leq n$, sequences $(x_{i_j}^j)_{i_j=1}^{m_j} \subset X_j$ such that $\|(x_{i_j}^j)_{i_j=1}^{m_j}\|_p^\omega \leq 1$. Using part (i), it is easy to see that the multiindex sequence $(x_{i_1}^1 \otimes \cdots \otimes x_{i_n}^n)_{i_1, \dots, i_n=1}^{m_1, \dots, m_n}$ in $X_1 \hat{\otimes}_{\alpha_p} \cdots \hat{\otimes}_{\alpha_p} X_n$, verifies

$$\|(x_{i_1}^1 \otimes \cdots \otimes x_{i_n}^n)_{i_1, \dots, i_n=1}^{m_1, \dots, m_n}\|_p^\omega \leq 1.$$

Using now that \bar{T} is p -summing, we obtain

$$\left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^p \right)^{1/p} \leq \pi_p(\bar{T}). \quad \square$$

Using this, we can give a simple proof of the following

Theorem 3.4. Let $n \geq 2$, let X_j be a $\mathcal{L}_{1, \lambda_j}$ space for $1 \leq j \leq n$, and let $1 \leq p \leq 2$. Then, every multilinear form $T: X_1 \times \cdots \times X_n \rightarrow \mathbb{K}$ is multiple p -summing and

$$\pi_p(T) \leq K_G^{2n-2} \prod_{j=1}^n \lambda_j \|T\|,$$

where K_G stands for the Grothendieck constant.

Proof. By standard localization procedures, all we have to do is to prove the result when $X_j = \ell_1^{k_j}$. We start showing the case $n = 2$. Using [7, Proposition 2.2], it is enough to see that, if q is the conjugate of p , $m_1, m_2 \in \mathbb{N}$, and $u_j: \ell_q^{m_j} \rightarrow \ell_1^{k_j}$ is a linear operator for $j = 1, 2$, then $S = T(u_1, u_2): \ell_q^{m_1} \times \ell_q^{m_2} \rightarrow \mathbb{K}$ verifies that

$$\pi_p(S) \leq K_G^2 \|u_1\| \|u_2\| \|T\|.$$

We know that the associated linear operator $S_1: \ell_q^{m_1} \rightarrow \ell_p^{m_2}$ can be factorized as $S_1 = u_2^* T_1 u_1$, where $T_1: \ell_1^{k_1} \rightarrow \ell_\infty^{k_2}$ is the linear operator associated to T . Grothendieck's theorem [7, Lemma 3.6] and [7, Theorem 3.11] tells us that $\pi_1(u_2^* T_1) \leq K_G^2 \|u_2\| \|T\|$. Therefore $\pi_p(S_1) \leq K_G^2 \|u_1\| \|u_2\| \|T\|$. Using Proposition 2.2 we are done.

To see the general case we reason by induction. We consider, for $j = 1, \dots, n$, natural numbers $m_j \in \mathbb{N}$ and linear operators $u_j: \ell_q^{m_j} \rightarrow \ell_1^{k_j}$. We have to show that $S = T(u_1, \dots, u_n): \ell_q^{m_1} \times \dots \times \ell_q^{m_n} \rightarrow \mathbb{K}$ verifies that

$$\pi_p(S) \leq K_G^{2n-2} \prod_{j=1}^n \|u_j\| \|T\|.$$

The associated linear operator $\bar{S}_{n-1}: \ell_q^{m_1} \otimes \dots \otimes \ell_q^{m_{n-1}} \rightarrow \ell_p^{m_n}$ can be factorized as $\bar{S}_{n-1} = u_n^* \bar{T}_{n-1}(u_1 \otimes \dots \otimes u_{n-1})$, where $\bar{T}_{n-1}: \ell_1^{k_1} \otimes \dots \otimes \ell_1^{k_{n-1}} \rightarrow \ell_\infty^{k_n}$ is the linear operator associated to T . Using the induction hypothesis and Proposition 3.1 we know that $\pi \leq K_G^{2(n-1)-2} \alpha_p = K_G^{2n-4} \alpha_p$ in $\ell_1^{k_1} \otimes \dots \otimes \ell_1^{k_{n-1}}$, where π denotes the projective tensor norm and α_p is the tensor norm defined at the beginning of this section. Therefore

$$\|u_1 \otimes \dots \otimes u_{n-1}\|_{\mathcal{L}(\ell_q^{m_1} \otimes_{\alpha_p} \dots \otimes_{\alpha_p} \ell_q^{m_{n-1}}, \ell_1^{k_1} \otimes_{\pi} \dots \otimes_{\pi} \ell_1^{k_{n-1}})} \leq K_G^{2n-4} \prod_{j=1}^{n-1} \|u_j\|.$$

As $\ell_1^{k_1} \otimes_{\pi} \dots \otimes_{\pi} \ell_1^{k_{n-1}} = \ell_1^{k_1 \dots k_{n-1}}$ and reasoning as in the bilinear case, we have that $u_n^* \bar{T}_{n-1}: \ell_1^{k_1} \otimes_{\pi} \dots \otimes_{\pi} \ell_1^{k_{n-1}} \rightarrow \ell_p^{m_n}$ verifies that $\pi_p(u_n^* \bar{T}_{n-1}) \leq K_G^2 \|u_n\| \|T\|$.

Therefore,

$$\pi_p(\bar{S}_{n-1}) \leq K_G^{2n-2} \prod_{j=1}^n \|u_j\| \|T\|.$$

Using Lemma 3.3 we get that $\pi_p(S_{n-1}) \leq K_G^{2n-2} \prod_{j=1}^n \|u_j\| \|T\|$, where S_{n-1} is the multilinear operator associated to \bar{S}_{n-1} and, finally, Proposition 2.2 tells us that $\pi_p(S) \leq \pi_p(S_{n-1})$, finishing the proof. \square

Remark 3.5. This result improves the results given in [3, Section 5]. It must be noticed that this theorem is essentially contained in the work of Rosenthal and Szarek [17]. However, the proof given here for our particular case is much shorter. Nevertheless, we have not been able to avoid the sharp reasonings of [17] to give a simpler proof of the following result, which is a straightforward corollary of [17, Theorem 1].

Theorem 3.6. *If $2 < p < \infty$, there exists a bilinear form $T: \ell_1 \times \ell_1 \rightarrow \mathbb{K}$ such that T is not multiple p -summing. In particular, there is not an inclusion theorem similar to [7, Theorem 2.8] for the class of multiple p -summing multilinear mappings.*

We want to improve now some of the results in [13]. With our approach, it is easy to understand why the results are true for precisely this values of p . We need some definitions first.

We recall that a multilinear operator $T: H_1 \times \dots \times H_n \rightarrow H$ defined on Hilbert spaces is said to be Hilbert–Schmidt if there exists $K > 0$ such that

$$\sum_{i_1, \dots, i_n=1}^{\infty} \|T(e_{i_1}^1, \dots, e_{i_n}^n)\|^2 < K, \quad (4)$$

where $(e_{ij}^j)_{ij} \subset H_j$ is an orthonormal basis ($1 \leq j \leq n$). In that case, the smallest K verifying (4) is the Hilbert–Schmidt norm of T . This class of operators was defined in [8] and studied and used in, for example, [5] or [15]. It is easy to see that T is Hilbert–Schmidt if and only if T is multiple 2-summing. Moreover, the multiple 2-summing norm and the Hilbert–Schmidt norm coincide [11, Proposition 5.5].

Using Proposition 2.2 it is easy to check that, for any $1 \leq p < \infty$, if T is Hilbert–Schmidt then T is multiple p -summing. Moreover, in [11] it is proved that, for any $2 \leq p < \infty$, T is Hilbert–Schmidt if and only if T is multiple p -summing.

A multilinear operator $T \in \mathcal{L}^k(X_1, \dots, X_k; Y)$ is said to be integral if there exists a regular Y^{**} -valued Borel measure G of bounded variation on the product $B_{X_1^*} \times \dots \times B_{X_k^*}$ such that

$$T(x_1, \dots, x_k) = \int_{B_{X_1^*} \times \dots \times B_{X_k^*}} x_1^*(x_1) \dots x_k^*(x_k) dG(x_1^*, \dots, x_k^*)$$

for all $(x_1, \dots, x_k) \in X_1 \times \dots \times X_k$. The space of integral multilinear operators $\mathcal{L}_I^k(X_1, \dots, X_k; Y)$ is a Banach space with the norm $\|T\|_{\text{int}} = \inf\{v(G), \text{ where } G \text{ represents } T \text{ as above}\}$.

These operators were defined in [19] (where they are called G -integral), although the definition is just a technical modification of a previous definition in [1].

We state for reference purposes the following proposition which we use later.

Proposition 3.7 [14, Corollary 3.2]. *Let X_j, Y_j , and Z be Banach spaces ($1 \leq j \leq n$). Let $u_j \in \Pi_2(X_j, Y_j)$ and $T \in \Pi_2^n(Y_1, \dots, Y_n; Z)$. Then $S = T(u_1, \dots, u_n)$ is integral.*

We can now prove a result, related to [13, Theorem 1].

Proposition 3.8. *If $T : \ell_1 \times \dots \times \ell_1 \rightarrow \ell_2$ verifies*

$$\sum_{i_1, \dots, i_n=1}^{\infty} \|T(e_{i_1}, \dots, e_{i_n})\|^2 < \infty,$$

then T is integral.

Proof. For simplicity in the notation we write the proof for $n = 2$. Let T be as in the hypothesis and let us call $K = (\sum_{i_1, i_2=1}^{\infty} \|T(e_{i_1}, e_{i_2})\|^2)^{1/2}$. Then, for any $(a, b) \in \ell_1 \times \ell_1$,

$$\|T(a, b)\| = \sum_{i_1, i_2=1}^{\infty} a_{i_1} b_{i_2} \|T(e_{i_1}, e_{i_2})\| \leq \|a\|_2 \|b\|_2 K$$

by Hölder's inequality. Therefore, we can extend T to a bilinear operator $\bar{T} : \ell_2 \times \ell_2 \rightarrow \ell_2$. Calling $i : \ell_1 \hookrightarrow \ell_2$ to the canonical inclusions, we have $T = \bar{T}(i, i)$. Since \bar{T} is Hilbert–Schmidt, it is multiple 2-summing.

According to one of the versions of Grothendieck's theorem [7, Theorem 1.13], the inclusion $i : \ell_1 \hookrightarrow \ell_2$ is 2-summing. To finish the proof, we just need to apply Proposition 3.7. \square

It is not difficult to prove by induction the following “multiple cotype inequality.”

Lemma 3.9. *Let Y be a Banach space with cotype $q < \infty$. Then, for any n -sequence $(y_{i_1, \dots, i_n})_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \subset Y$, the following inequality holds:*

$$\left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|y_{i_1, \dots, i_n}\|^q \right)^{1/q} \leq c_q(Y)^n \left(\int_0^1 \dots \int_0^1 \left\| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} y_{i_1, \dots, i_n} r_{i_1}(t_1) \dots r_{i_n}(t_n) \right\|^q dt_1 \dots dt_n \right)^{1/q}.$$

With this inequality, we can prove a result relating r -dominated operators with multiple q -summing operators.

Theorem 3.10. *Let $1 \leq r, q < \infty$ and let $T : X_1 \times \dots \times X_n \rightarrow Y$ be an r -dominated multilinear operator. If $r \leq q$ then T is multiple q -summing and*

$$\pi_q(T) \leq \delta_r(T).$$

If $q < r$ and Y has cotype q , then T is multiple $(q, 2)$ -summing and

$$\pi_{(q,2)}(T) \leq c_q(Y)^n B_r^n \delta_r(T),$$

where B_r is the constant appearing in Khinchin’s inequality [7, Theorem 1.10]. In particular, r -dominated multilinear forms are always multiple 2-summing.

Proof. If $r \leq q$, the result follows immediately from Theorem 1.1. So, let us suppose $q < r$.

Let $(x_{i_j}^j)_{i_j=1}^{m_j} \subset X_j$. Then, using Lemma 3.9, Theorem 1.1, and Khinchin’s inequality, we have

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^q \right)^{1/q} \\ & \leq c_q(Y)^n \left(\int_0^1 \dots \int_0^1 \left\| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} T(x_{i_1}^1, \dots, x_{i_n}^n) r_{i_1}(t_1) \dots r_{i_n}(t_n) \right\|^q dt_n \dots dt_1 \right)^{1/q} \\ & \leq c_q(Y)^n \left(\int_0^1 \dots \int_0^1 \left\| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} T(x_{i_1}^1, \dots, x_{i_n}^n) r_{i_1}(t_1) \dots r_{i_n}(t_n) \right\|^r dt_n \dots dt_1 \right)^{1/r} \\ & \leq c_q(Y)^n \delta_r(T) \left(\int_0^1 \dots \int_0^1 \prod_{j=1}^n \int_{B_{X_j}^*} \left| \sum_{i_j=1}^{m_j} x_j^*(x_{i_j}^j) r_{i_j}(t_j) \right|^r d\mu_j(x_j^*) dt_n \dots dt_1 \right)^{1/r} \end{aligned}$$

$$\begin{aligned}
&= c_q(Y)^n \delta_r(T) \prod_{j=1}^n \left(\int_{B_{X_j^*}} \int_0^1 \left| \sum_{i_j=1}^{m_j} x_j^*(x_{i_j}^j) r_{i_j}(t_j) \right|^r dt_j d\mu_j(x_j^*) \right)^{1/r} \\
&\leq c_q(Y)^n B_r^n \delta_r(T) \prod_{j=1}^n \left(\int_{B_{X_j^*}} \left(\sum_{i_j=1}^{m_j} |x_j^*(x_{i_j}^j)|^2 \right)^{r/2} d\mu_j(x_j^*) \right)^{1/r} \\
&\leq c_q(Y)^n B_r^n \delta_r(T) \prod_{j=1}^n \| (x_{i_j}^j)_{i_j=1}^{m_j} \|_2^\omega. \quad \square
\end{aligned}$$

Remarks 3.11. Some comments are in order:

- (1) Compare this result with [10, Theorem 3.1]. Although none of the results follows from the other one, in a sense Theorem 3.10 is a big improvement of [10, Theorem 3.1].
- (2) The converse of Theorem 3.10 is not true: according to [3, Theorem 3.1], every multilinear form from the product of \mathcal{L}_∞ spaces is multiple 2-summing. Yet, there exist trilinear forms $T: \ell_\infty \times \ell_\infty \times \ell_\infty \rightarrow \mathbb{K}$ such that their associated linear operator $T_1: \ell_\infty \rightarrow \mathcal{L}^2(\ell_\infty, \ell_\infty)$ is not weakly compact [2], hence they cannot be r -dominated.
- (3) A tempting improvement of Theorem 3.10 stating, for instance, that r -dominated multilinear forms are multiple p -summing for every $p \in [1, \infty)$ is false: according to Grothendieck's theorem, every bilinear form on $C(K) \times C(K)$ is 2-dominated; yet, since the exponent in Littlewood's inequality is optimal [6, Proposition 34.11], for every $p \in [1, 4/3)$ there exist bilinear forms on $c_0 \times c_0$ which are not multiple p -summing.
- (4) The referee kindly pointed to us that the case $r \leq q$ of Theorem 3.10 appeared in [11], and the case $q < r$ has been recently (and independently) obtained in [18].

We can now prove a substantial improvement of [13, Theorem 3].

Corollary 3.12. Let $2 \leq p, q < \infty$ and $r > q$ (the case $r \leq q$ is much easier). Let Y be a Banach space of cotype q . If $T: \ell_p \times \cdots \times \ell_p \rightarrow Y$ is an r -dominated n -linear operator, then

$$\left(\sum_{i_1, \dots, i_n=1}^{\infty} \|T(e_{i_1}, \dots, e_{i_n})\|^q \right)^{1/q} \leq (c_q(Y) B_r)^n \delta_r(T).$$

In particular, if $T: \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$ is an r -dominated n -linear form, then

$$\left(\sum_{i_1, \dots, i_n=1}^{\infty} |T(e_{i_1}, \dots, e_{i_n})|^2 \right)^{1/2} \leq (B_r)^n \delta_r(T).$$

Proof. According to Theorem 3.10, T is multiple $(q, 2)$ -summing and verifies that $\pi_{(q,2)}(T) \leq (c_q(Y)B_r)^n \delta_r(T)$. As $\|(e_i)_{i=1}^\infty\|_2^\omega \leq 1$ in ℓ_p , we get that

$$\left(\sum_{i_1, \dots, i_n=1}^\infty \|T(e_{i_1}, \dots, e_{i_n})\|^q \right)^{1/q} \leq (c_q(Y)B_r)^n \delta_r(T). \quad \square$$

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